

# ON THE REGULARITY INDEX OF $s$ EQUIMULTIPLE FAT POINTS NOT ON A LINEAR $(r - 1)$ -SPACE, $s \leq r + 3$

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## Abstract

We prove the Trung's conjecture about Segre's upper bound for  $s$  equimultiple fat points not on a linear  $(r - 1)$ -space,  $s \leq r + 3$ , by algebraic method used in [3]. This method also may used to research other cases of fat points.

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## 1 Introduction

Let  $K$  be an algebraically closed of arbitrary characteristic and let  $P_1, \dots, P_s$  be distinct points in the projective space  $\mathbb{P}^n := \mathbb{P}_K^n$ . Denote by  $\wp_1, \dots, \wp_s$  the homogeneous prime ideals of the polynomial ring  $R := K[X_0, \dots, X_n]$  corresponding to the points  $P_1, \dots, P_s$ .

Let  $m_1, \dots, m_s$  be positive integers. Denote by  $m_1P_1 + \dots + m_sP_s$  the zero-scheme defined by the ideal  $I := \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$ . Then

$$Z := m_1P_1 + \dots + m_sP_s$$

is called to be a set of fat points in  $\mathbb{P}^n$ .

The ring  $R/I$  is the homogeneous coordinate ring of  $Z$ . It is a graded ring,  $R/I = \bigoplus_{t \geq 0} (R/I)_t$ , whose multiplicity is  $e(R/I) := \sum_{i=1}^s \binom{m_i+n-1}{n}$ . We also call it to be the multiplicity of  $Z$ , and denote it by  $\text{reg}(Z)$ .

The Hilbert function of  $Z$  is defined to be  $H_Z(t) := \dim_K (R/I)_t$ . This function strictly increases until it reaches  $e(Z)$ , at which it stabilizes. Then the number

$$\min\{t \in \mathbb{Z} | H_A(t) = e(A)\}$$

is called the regularity index of  $Z$ , and denote it by  $\text{reg}(Z)$ . It is well known that  $\text{reg}(Z) = \text{reg}(R/I)$ , the Castelnuovo-Mumford regularity of  $R/I$ .

The problem to exactly determine  $\text{reg}(Z)$  is more fairly difficult. So, instead of it one to find an upper bound for  $\text{reg}(Z)$ .

For generic fat points  $Z = m_1P_1 + \dots + m_sP_s$  in  $\mathbb{P}^2$  with  $m_1 \geq \dots \geq m_s$ , Segre [6] showed that

$$\text{reg}(Z) \leq \max \left\{ m_1 + m_2 - 1, \left\lceil \frac{m_1 + \dots + m_s}{2} \right\rceil \right\}.$$

A set of fat points  $Z = m_1P_1 + \dots + m_sP_s$  in  $\mathbb{P}^n$  is said to be in general position if no  $j + 2$  of the points  $P_1, \dots, P_s$  are on any  $j$ -plane for  $j < n$ . A set of generic fat points always in general position. Segre's upper bound later was generalised by Catalisano, Trung, Valla [3] for fat points in general position in  $\mathbb{P}^n$

$$\text{reg}(Z) \leq \max \left\{ m_1 + m_2 - 1, \left\lceil \left( \sum_{i=1}^s m_i + n - 2 \right) / n \right\rceil \right\}.$$

In 1996, Trung conjectured an upper bound for the regularity index of arbitrary fat points  $Z = m_1P_1 + \cdots + m_sP_s$  in  $\mathbb{P}^n$  (see [8], [9]).

**Conjecture.**

$$\text{reg}(Z) \leq \max\{T_j \mid j = 1, \dots, n\},$$

where

$$T_j = \max \left\{ \left\lfloor \frac{\sum_{l=1}^q m_{i_l} + j - 2}{j} \right\rfloor \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a linear } j\text{-space} \right\}.$$

He called this upper bound to be the Segre's upper bound. This upper bound nowadays referred to as Segre's bound.

The same conjecture was also given independently by Fatabbi and Lorenzini (see [4], [5]).

The Trung's conjecture has been proved in many cases:  $n = 2$  (see [4], [7]),  $n = 3$  (see [5], [8]),  $n = 4$  and  $m_1 = \cdots = m_s = 2$  (see [9]), for  $n + 2$  non-degenerate fat points in  $\mathbb{P}^n$  (see [2]), for  $s + 2$  fat points whose support not on a linear  $(s - 1)$ -space (see [10]), for  $n + 3$  non-degenerate almost equimultiple fat points in  $\mathbb{P}^n$  (see [12]), and recently for  $n + 3$  non-degenerate fat points in  $\mathbb{P}^n$  (see [12]).

A set of  $s$  points  $P_1, \dots, P_s$  in  $\mathbb{P}^n$  is said to be in general position on a linear  $r$ -space  $\alpha$  if all points  $P_1, \dots, P_s$  lie on the  $\alpha$  and no  $j + 2$  of these points lie on a linear  $j$ -space for  $j < r$ . So if  $r = n$ , then we get the case of the points which are in general position in  $\mathbb{P}^n$ .

A set of  $s$  fat points  $Z = m_1P_1 + \cdots + m_sP_s$  in  $\mathbb{P}^n$  is said to be equimultiple if  $m_1 = \cdots = m_s$ .

In this paper by algebraic method used in [3], we prove the Trung's conjecture about Segre's upper bound for  $s$  equimultiple fat points not on a linear  $(r - 1)$ -space,  $s \leq r + 3$ . This method also may used to research other cases of fat points.

## 2 Preliminaries

From now on, we say a  $j$ -plane, i.e. a linear  $j$ -space. We identify a hyperplane as the linear form defining it.

We use the following lemmas which have been proved in [3], [10]. The first lemma allows us to compute the regularity index by induction.

**Lemma 2.1.** [3, Lemma 1] *Let  $P_1, \dots, P_r, P$  be distinct points in  $\mathbb{P}^n$  and let  $\wp$  be the defining ideal of  $P$ . If  $m_1, \dots, m_r$  and  $a$  are positive integers,  $J := \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r}$ , and  $I = J \cap \wp^a$ , then*

$$\text{reg}(R/I) = \max\{a - 1, \text{reg}(R/J), \text{reg}(R/(J + \wp^a))\}.$$

To estimate  $\text{reg}(R/(J + \wp^a))$  we shall use the following lemma.

**Lemma 2.2.** [3, Lemma 3] *Let  $P_1, \dots, P_r$  be distinct points in  $\mathbb{P}^n$  and  $a, m_1, \dots, m_r$  positive integers. Put  $J = \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r}$  and  $\wp = (X_1, \dots, X_n)$ . Then*

$$\text{reg}(R/(J + \wp^a)) \leq b$$

*if and only if  $X_0^{b-i}M \in J + \wp^{i+1}$  for every monomial  $M$  of degree  $i$  in  $X_1, \dots, X_n$ ,  $i = 0, \dots, a - 1$ .*

Suppose that we can find  $t$  hyperplanes  $L_1, \dots, L_t$  avoiding  $P$  such that  $L_1 \cdots L_t M \in J$ . For  $j = 1, \dots, t$ , since we can write  $L_j = X_0 + G_j$  for some linear form  $G_j \in \wp$ , we get  $X_0^t M \in J + \wp^{t+1}$ . Therefore, we have the following remark:

**Remark 2.3.** *Assume that  $L_1, \dots, L_t$  are hyperplanes avoiding  $P$  such that  $L_1 \cdots L_t M \in J$  for every monomial  $M$  of degree  $i$  in  $X_1, \dots, X_n$ ,  $i = 0, \dots, a - 1$ . If*

$$\delta = \max\{t + i \mid 0 \leq i \leq a - 1\},$$

then

$$\text{reg}(R/(J + \wp^a)) \leq \delta.$$

The following lemma are the main results of [10].

**Lemma 2.4.** [10, Theorem 3.4] *Let  $P_1, \dots, P_{s+2}$  be distinct points not on a linear  $(s-1)$ -space in  $\mathbb{P}^n$ ,  $s \leq n$ , and  $m_1, \dots, m_{s+2}$  be positive integers. Put  $I = \wp_1^{m_1} \cap \dots \cap \wp_{s+2}^{m_{s+2}}$ ,  $A = R/I$ . Then,*

$$\text{reg}(A) = \max\{T_j \mid j = 1, \dots, n\},$$

where

$$T_j = \max \left\{ \left\lfloor \frac{\sum_{i=1}^q m_{i_l} + j - 2}{j} \right\rfloor \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a linear } j\text{-space} \right\},$$

$j = 1, \dots, n$ .

### 3 Regularity index of $s$ equimultiple fat points not on a linear $(r-1)$ -space, $s \leq r+3$

The following lemma help us to find a sharp upper bound for the regularity index of  $s$  fat points in  $\mathbb{P}^n$ .

**Lemma 3.1.** *Let  $P_1, \dots, P_s, P$  be distinct points in  $\mathbb{P}^n$  such that for  $r$  arbitrary points of  $\{P_1, \dots, P_s\}$ , there always exists a linear  $(r-1)$ -space passing through these  $r$  points and avoiding  $P$ . Let  $m_1, \dots, m_s$  be positive integers. Consider the set  $\{P_1, \dots, P_s\}$  with the chain of multiplicities  $(m_1, \dots, m_s)$ . Assume that  $t$  is an integer such that*

$$t \geq \max \left\{ m_j, \left\lfloor \frac{\sum_{i=1}^s m_i + r - 1}{r} \right\rfloor \mid j = 1, \dots, s \right\}.$$

*Then, there exist  $t$  linear  $(r-1)$ -spaces, say  $L_1, \dots, L_t$ , avoiding  $P$  such that for every point  $P_j \in \{P_1, \dots, P_s\}$ , there are  $m_j$  linear  $(r-1)$ -spaces (including multiplicity) of  $\{L_1, \dots, L_t\}$  passing through the  $P_j$ .*

*Proof.* We argue by induction on  $\sum_{i=1}^s m_i$ . We may assume that  $m_1 \geq \dots \geq m_s$  (after relabelling, if necessary). If  $s \leq r$ , then by the assumption there is a linear  $(r-1)$ -space, say  $L$ , passing through all points  $P_1, \dots, P_s$  and avoiding  $P$ . Let  $L_1 = \dots = L_t = L$ . Since  $t \geq \max\{m_j \mid j = 1, \dots, s\}$ , for every point  $P_j$  there exists  $m_j$  linear  $(r-1)$ -spaces of  $\{L_1, \dots, L_t\}$  passing through the  $P_j$ .

If  $s > r$ , then by the assumption there is a linear  $(r-1)$ -space, say  $L_1$ , passing through all points  $P_1, \dots, P_r$  and avoiding  $P$ . Since  $t \geq \left\lfloor \frac{\sum_{i=1}^s m_i + r - 1}{r} \right\rfloor$ , we have

$$t - 1 \geq \left\lfloor \frac{(m_1 - 1) + \dots + (m_r - 1) + m_{r+1} + \dots + m_s + r - 1}{r} \right\rfloor.$$

On the other hand, since  $t \geq \left\lfloor \frac{\sum_{i=1}^s m_i + r - 1}{r} \right\rfloor$  and  $m_1 \geq m_2 \geq \dots \geq m_s$ , we have

$$t - 1 \geq \left\lfloor \frac{(r+1)m_{r+1} + r - 1}{r} \right\rfloor - 1 \geq m_{r+1}.$$

So,

$$t - 1 \geq \max\{m_1 - 1, \dots, m_r - 1, m_{r+1}, \dots, m_s\}.$$

Consider the set  $\{P_1, \dots, P_s\}$  with the chain of multiplicities  $(m_1 - 1, \dots, m_r - 1, m_{r+1}, \dots, m_s)$ . By inductive assumption we can find  $t - 1$  linear  $(r-1)$ -spaces, say  $L_2, \dots, L_t$ , avoiding  $P$  such that for  $j = 1, \dots, r$  there are  $m_j - 1$  linear  $(r-1)$ -spaces of  $\{L_1, \dots, L_t\}$  passing through the point  $P_j$ ; for  $j = r + 1, \dots, s$  there are  $m_j$  linear  $(r-1)$ -spaces of  $\{L_1, \dots, L_t\}$  passing through the point  $P_j$ . Therefore, we have  $t$  linear  $(r-1)$ -space  $L_1, \dots, L_t$  as desired.  $\square$

**Lemma 3.2.** *Let  $X = \{P_1, \dots, P_{s+3}\}$  be a set of distinct points lie on a linear  $s$ -space in  $\mathbb{P}^n$ ,  $3 \leq s \leq n$ , such that there is not any linear  $(s-1)$ -space containing  $s+2$  points of  $X$  and there is not any linear  $(s-2)$ -space containing  $s$  points of  $X$ . Let  $\wp_1, \dots, \wp_{s+3}$  be the homogeneous prime ideals of the polynomial ring  $R = K[X_0, \dots, X_n]$  corresponding to the points  $P_1, \dots, P_s$ . Assume that there is a linear  $(s-1)$ -space, say  $\alpha$ , containing  $s+1$  points  $P_1, \dots, P_{s+1}$  and there is a linear  $(s-1)$ -space, say  $\beta$ , containing  $s+1$  points  $P_3, \dots, P_{s+3}$ . Let  $m$  be a positive integer. For  $j = 1, \dots, n$ , put*

$$T_j = \max \left\{ \left\lceil \frac{mq + j - 2}{j} \right\rceil \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a linear } j\text{-space} \right\}.$$

Then,

$$\text{reg}(R/(J + \wp_{s+3}^m)) \leq \max\{T_j \mid j = 1, \dots, n\},$$

where  $J = \wp_1^m \cap \dots \cap \wp_{s+2}^m$ .

*Proof.* We remark that there exists a linear  $(s-1)$ -space, say  $\gamma$ , containing  $P_1, P_2, P_{s+2}, s-3$  points of  $\{P_3, \dots, P_{s+1}\}$  and avoiding  $P_{s+3}$ . In fact, we assume that  $\pi$  is a linear  $(s-1)$ -space containing  $P_1, P_2, P_{s+2}, s-3$  points  $P_5, \dots, P_{s+1}$  and  $P_{s+3}$ . Then let  $\gamma$  be the linear  $(s-1)$ -space containing  $P_1, P_2, P_{s+2}, P_4, P_5, \dots, P_s$ . We have  $P_{s+3} \notin \gamma$  (If  $P_{s+3} \in \gamma$ , then  $\gamma = \pi$  is a linear  $(s-1)$ -space containing  $s+2$  points  $P_1, P_2, P_4, \dots, P_{s+3}$  of  $X$ , a contradiction).

We may assume that  $P_1, P_2, P_4, P_5, \dots, P_s, P_{s+2} \in \gamma$ . Since arbitrary  $s$  points of  $\beta \cap X$  do not lie on a linear  $(s-2)$ -space, we can put  $P_{s+3} = (1, 0, \dots, 0)$ ,  $P_3 = (0, 0, 1, 0, \dots, 0)$ ,  $P_5 = (0, 0, 0, \underbrace{1}_4, 0, \dots, 0)$ , ...,  $P_j = (0, \dots, 0, \underbrace{1}_{j-1}, 0, \dots, 0)$ ,  $j = 5, \dots, s+2$ . Since  $P_2 \notin \beta$ , we can put

$$P_2 = (0, 1, 0, \dots, 0).$$

For every monomial  $M = X_1^{c_1} \dots X_n^{c_n}$ ,  $c_1 + \dots + c_n = i$ ,  $i = 0, \dots, m-1$ . Put  $m_1 = m_4 = m$ ,  $m_2 = m - i + c_1$ ,  $m_3 = m - i + c_2$ ,  $m_j = m - i + c_{j-2}$ ,  $j = 5, \dots, s+2$ . Let  $H$  be a hyperplane containing  $\alpha$  and avoiding  $P_{s+3}$ , let  $L$  be a hyperplane containing  $\gamma$  and avoiding  $P_{s+3}$ . Put

$$t = \max\{m_3, m_{s+1}\}.$$

Since  $c_s + \max\{c_2, c_{s-1}\} \leq i$ , we have  $m_{s+2} + t \leq 2m - i$ . We consider the following cases:

**Case 1:**  $m_{s+2} + t \leq 2m - i - 1$ , or  $s = 3$ , or  $s = 4$  and  $m = 2$ . We have  $P_1, P_2, P_4, \dots, P_s \in H \cap L$ ;  $P_3, P_{s+1} \in H$ ;  $P_{s+2} \in L$ . Therefore,

$$H^{\max\{m-m_{s+2}, t\}} L^{m_{s+2}} \in \wp_1^m \cap \wp_2^m \cap \wp_3^t \cap \wp_4^m \cap \dots \cap \wp_s^m \cap \wp_{s+1}^t \cap \wp_{s+2}^{m_{s+2}}.$$

Moreover, since  $M \in \wp_3^{m-m_3} \cap \wp_{s+1}^{m-m_{s+1}} \cap \wp_{s+2}^{m-m_{s+2}}$  and  $t = \max\{m_3, m_{s+1}\}$ , we have

$$H^{\max\{m-m_{s+2}, t\}} L^{m_{s+2}} M \in \wp_1^m \cap \dots \cap \wp_{s+2}^m = J.$$

By Remark 2.3 we get

$$\begin{aligned} \text{reg}(R/(J + \wp_{s+3}^m)) &\leq \max\{\max\{m - m_{s+2}, t\} + m_{s+2} + i \mid i = 1, \dots, m-1\} \\ &\leq \max\{m + i, t + m_{s+2} + i \mid i = 0, \dots, m-1\}. \end{aligned}$$

If  $m_{s+2} + t \leq 2m - i - 1$ , then

$$\max\{m + i, t + m_{s+2} + i \mid i = 0, \dots, m-1\} \leq 2m - 1 = T_1.$$

If  $s = 3$ , then

$$\max\{m + i, t + m_{s+2} + i \mid i = 0, \dots, m-1\} \leq 2m = T_2.$$

If  $s = 4$  and  $m = 2$ , then

$$\max\{m + i, t + m_{s+2} + i \mid i = 0, \dots, m-1\} \leq 2m = 4 = T_4.$$

**Case 2:**  $m_{s+2} + t = 2m - i$  and  $s \geq 4$  and  $m \geq 3$ , or  $m_{s+2} + t = 2m - i$  and  $s \geq 5$  and  $m = 2$ . Without loss of generality we can assume that  $m_3 \geq m_{s+1}$ . Then  $t = \max\{m_3, m_{s+1}\} = m_3$ . We

have  $2m - i = m_{s+2} + t = m_{s+2} + m_3 = m - i + c_s + m - i + c_2$ . This implies  $c_2 + c_s = i$ . So,  $c_j = 0$ , for every  $2 \neq j \neq s$ . We consider two following cases for  $i$ :

**Case 2.1:**  $i = 0$ . By the assumption, there is not any linear  $(s - 2)$ -space containing  $s$  points of  $X$ , we have  $P_3, P_{s+1}, P_{s+2}$  and  $P_{s+3}$  not on a linear 2-space. Therefore, there is a hyperplane, say  $\sigma$ , containing  $P_3, P_{s+1}, P_{s+2}$  and avoiding  $P_{s+3}$ . Recall that  $P_1, \dots, P_{s+1} \in H$  and  $P_1, P_2, P_4, \dots, P_s, P_{s+2} \in L$ . Thus, we have

$$H^{m-1}L^{m-1}\sigma \in \wp_1^m \cap \dots \cap \wp_{s+2}^m = J.$$

It follows that

$$H^{m-1}L^{m-1}\sigma M \in J \quad (1)$$

with  $m - 1 + m - 1 + 1 + i = 2m - 1 = T_1$ .

**Case 2.2:**  $i \geq 1$ . We consider three following cases for  $c_s$ :

- $c_s < i$ : Since  $P_3, \dots, P_{s+3}$  lie on the linear  $(s - 1)$ -space  $\beta$  and there is not any linear  $(s - 2)$ -space containing  $s$  points of  $X$ , the linear  $(s - 1)$ -space containing  $s$  points  $P_1, P_3, \dots, P_s, P_{s+2}$  avoids  $P_{s+3}$ . Then there exists a hyperplane, say  $\pi$ , containing  $P_1, P_3, \dots, P_s, P_{s+2}$  and avoiding  $P_{s+3}$ . Moreover, since  $P_1, \dots, P_{s+1} \in H$  and  $P_1, P_2, P_4, \dots, P_s, P_{s+2} \in L$ , we have

$$H^{m_3-1}L^{m_{s+2}-1}\pi \in \wp_1^{m_{s+2}+m_3-1} \cap \wp_2^{m_{s+2}+m_3-2} \cap \wp_3^{m_3} \cap \wp_4^{m_{s+2}+m_3-1} \cap \dots \cap \wp_s^{m_{s+2}+m_3-1} \cap \wp_{s+1}^{m_3-1} \cap \wp_{s+2}^{m_{s+2}}.$$

Since  $m_{s+2} + m_3 = 2m - i$  and  $i \leq m - 1$ , we have  $m_{s+2} + m_3 - 1 = 2m - i - 1 \geq m$ . Since  $i \geq 1$ , we have  $m_2 = m - i \leq m - 1 \leq m_{s+2} + m_3 - 2$ . Since  $c_2 + c_s = i$  and  $c_s < i$ , we have  $c_2 \geq 1$ . Note that  $c_{s-1} = 0$ . Thus,  $m_{s+1} = m - i + c_{s-1} \leq m - i + c_2 = m_3 - 1$ . Therefore, we have

$$H^{m_3-1}L^{m_{s+2}-1}\pi \in \wp_1^m \cap \wp_2^{m-i} \cap \wp_3^{m-i+c_2} \cap \wp_4^m \cap \dots \cap \wp_s^m \cap \wp_{s+1}^{m-i+c_{s-1}} \cap \wp_{s+2}^{m-i+c_s}.$$

Note that  $c_j = 0$  for every  $2 \neq j \neq s$ . So,  $M \in \wp_2^i \cap \wp_3^{i-c_2} \cap \wp_5^i \cap \dots \cap \wp_s^i \cap \wp_{s+1}^{i-c_{s-1}} \cap \wp_{s+2}^{i-c_s}$ . Therefore,

$$H^{m_3-1}L^{m_{s+2}-1}\pi M \in \wp_1^m \cap \dots \cap \wp_{s+2}^m = J \quad (2)$$

with  $m_3 - 1 + m_{s+2} - 1 + 1 + i = 2m - i - 1 + i = 2m - 1 = T_1$ .

- $c_s = i$  and  $m \geq 3$ : Then we have  $c_j = 0, j = 1, \dots, s - 1$ . Since  $P_1, P_3, P_4, \dots, P_s, P_{s+2}, P_{s+3}$  do not lie on a linear  $(s - 1)$ -space, there exists a hyperplane, say  $\sigma_1$ , containing  $P_1, P_3, P_4, \dots, P_s, P_{s+2}$  and avoiding  $P_{s+3}$  (If  $P_1, P_3, P_4, \dots, P_s, P_{s+2}, P_{s+3}$  lie on a linear  $(s - 1)$ -space, then this linear  $(s - 1)$ -space contains  $P_{s+1}$ . It follows that there is a linear  $(s - 1)$ -space containing  $s + 2$  points of  $X$ , a contradiction). Similarly, since  $P_1, P_4, \dots, P_s, P_{s+1}, P_{s+2}, P_{s+3}$  do not lie on a linear  $(s - 1)$ -space, there exists a hyperplane, say  $\sigma_2$ , containing  $P_1, P_4, \dots, P_s, P_{s+1}, P_{s+2}$  and avoiding  $P_{s+3}$ . Moreover, since  $P_1, \dots, P_{s+1} \in H$  and  $P_1, P_2, P_4, \dots, P_s, P_{s+2} \in L$ , we have

$$H^{m-i-1}L^{m-2}\sigma_2\sigma_1 \in \wp_1^{2m-i-1} \cap \wp_2^{m-i} \cap \wp_3^{m-i} \cap \wp_4^{2m-i-1} \cap \dots \cap \wp_s^{2m-i-1} \cap \wp_{s+1}^{m-i} \cap \wp_{s+2}^m.$$

Since  $2m - i - 1 \geq m$ , we have

$$H^{m-i-1}L^{m-2}\sigma_2\sigma_1 \in \wp_1^m \cap \wp_2^{m-i} \cap \wp_3^{m-i} \cap \wp_4^m \cap \dots \cap \wp_s^m \cap \wp_{s+1}^{m-i} \cap \wp_{s+2}^m.$$

Moreover, since  $M \in \wp_2^i \cap \wp_3^i \cap \wp_5^i \cap \dots \cap \wp_s^i \cap \wp_{s+1}^i$ , we have

$$H^{m-i-1}L^{m-2}\sigma_2\sigma_1 M \in \wp_1^m \cap \dots \cap \wp_{s+2}^m = J \quad (3)$$

with  $m - i - 1 + m - 2 + 2 + i = 2m - 1 = T_1$ .

- $c_s = i$  and  $m = 2$  and  $s \geq 5$ . Since  $m = 2$  and  $1 \leq i \leq m - 1$ , we get  $i = 1$ . Let  $\beta_1$  be a linear  $(s - 1)$ -space containing  $s$  points  $P_1, P_3, \dots, P_{s-1}, P_{s+1}, P_{s+2}$ . If  $P_{s+3} \in \beta_1$ , then  $s$  points  $P_3, \dots, P_{s-1}, P_{s+1}, P_{s+2}$  are contained in the linear  $(s - 2)$ -space  $\beta_1 \cap \beta$ . This contradicts our

assumption. So,  $P_3 \notin \beta_1$ . Therefore, there exists a hyperplane, say  $\varrho$ , containing  $\beta_1$  and avoiding  $P_{s+3}$ . Recall that  $L$  contains  $P_1, P_2, P_4, \dots, P_s, P_{s+2}$ . Thus, we have

$$\varrho L \in \wp_1^2 \cap \wp_2 \cap \wp_3 \cap \wp_4^2 \cap \wp_5 \cap \dots \cap \wp_{s+1} \cap \wp_{s+2}^2.$$

Moreover, since  $M \in \wp_2 \cap \wp_3 \cap \wp_5 \cap \dots \cap \wp_s \cap \wp_{s+1}$ , we have

$$\varrho LM \in \wp_1^2 \cap \dots \cap \wp_{s+2}^2 = J \quad (4)$$

with  $1 + 1 + i = 3 = T_1$ .

From (1), (2), (3), (4) and by Remark 2.3 we get

$$\text{reg}(R/(J + \wp_{s+3}^m)) \leq T_1.$$

□

We need the following proposition to find a sharp upper bound for the regularity index of  $s + 3$  equimultiple fat points not on a linear  $(s - 1)$ -space.

**Proposition 3.3.** *Let  $X = \{P_1, \dots, P_{s+3}\}$  be a set of distinct points lie on a linear  $s$ -space  $\gamma$  but  $X$  is not in general position on  $\gamma$  and  $X$  does not lie on a linear  $(s - 1)$ -space in  $\mathbb{P}^n$ ,  $2 \leq s \leq n$ . Let  $m$  be a positive integer. Assume that  $\wp_1, \dots, \wp_{s+3}$  are the homogeneous prime ideals of the polynomial ring  $R = K[X_0, \dots, X_n]$  corresponding to the points  $P_1, \dots, P_{s+3}$ . For  $j = 1, \dots, n$ , put*

$$T_j = \max \left\{ \left\lceil \frac{mq + j - 2}{j} \right\rceil \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a linear } j\text{-space} \right\}.$$

Then, there exists a point  $P_{i_0} \in X$  such that

$$\text{reg}(R/(J + \wp_{i_0}^m)) \leq \max\{T_j \mid j = 1, \dots, n\},$$

where

$$J = \bigcap_{i \neq i_0} \wp_i^m.$$

*Proof.* We have  $T_1 \geq 2m - 1$  and  $T_s = \left\lceil \frac{(s+3)m+s-2}{s} \right\rceil \geq T_{s+1} \geq \dots \geq T_n$ . We consider two following cases:

**Case 1:** There exists a hyperplane, say  $H$ , avoiding a point of  $X$  and passing through  $s + 2$  points of  $X$ . We may assume that  $P_{s+3} \notin H$  and  $P_1, \dots, P_{s+2} \in H$  (after relabeling, if necessary). Put  $P_{i_0} = P_{s+3} = (1, 0, \dots, 0)$ . For every monomial  $M = X_1^{c_1} \dots X_n^{c_n}$ ,  $c_1 + \dots + c_n = i$ ,  $i = 0, \dots, m - 1$ . We have  $H^m \in J$ . It implies that  $H^m M \in J$ . By Remark 2.3 we get

$$\text{reg}(R/(J + \wp_{i_0}^m)) \leq \max\{m + i \mid i = 0, \dots, m - 1\} \leq 2m - 1 \leq T_1.$$

**Case 2:** There does not exist any hyperplane avoiding a point of  $X$  and passing through  $s + 2$  points of  $X$ . This implies that there does not exist any linear  $(s - 1)$ -space passing through  $s + 2$  points of  $X$ . So, a linear  $(s - 1)$ -space contains at most  $s + 1$  points of  $X$ . Since  $X$  is not in general position in the linear  $s$ -space  $\gamma$ , there exists a linear  $(s - 1)$ -space containing  $s + 1$  points of  $X$ . Put

$$k := \min\{h \mid \text{there exists a linear } h\text{-space containing } h + 2 \text{ points of } X\}.$$

Then,  $k \leq s - 1$ . Since a linear  $(s - 1)$ -space containing at most  $s + 1$  points of  $X$ , we have a linear  $h$ -space containing at most  $h + 2$  points of  $X$  with  $h \leq s - 1$ . Thus,  $T_k \geq T_h$ ,  $h = k + 1, \dots, s - 1$ .

Let  $\alpha$  be a linear  $k$ -space containing  $k + 2$  points of  $X$ . We may assume that  $P_1, \dots, P_{k+2} \in \alpha$  and  $P_{k+3}, \dots, P_{s+3} \notin \alpha$  (after relabeling, if necessary). We consider the two following subcases:

**Case 2.1:**  $m = 1$  or  $s - k \geq k$  or  $s - k \geq 2$ . For arbitrary  $s$  points of  $X$ , there always exists a linear  $(s - 1)$ -space containing them. Thus, there is a linear  $(s - 1)$ -space, say  $\beta$ , such that  $\beta$  contains

$P_{k+3}, \dots, P_{s+3}$  and  $\beta$  contains  $k-1$  points of  $X \cap \alpha$ . We are considering Case 2, so  $\beta$  avoids two points of  $X \cap \alpha$ . We may assume that  $P_4, \dots, P_{k+2} \in \beta$ ,  $P_1 \notin \beta$ ,  $P_2 \notin \beta$ .

By the property of  $k$ , we have arbitrary  $k+1$  points of  $X \cap \alpha$  not on a linear  $(k-1)$ -space. We can put  $P_{i_0} = P_1 = (1, 0, \dots, 0)$ ,  $P_2 = (0, \underbrace{1}_2, 0, \dots, 0)$ ,  $P_3 = (0, 0, \underbrace{1}_3, 0, \dots, 0)$ ,  $P_5 = (0, 0, 0, \underbrace{1}_4, 0, \dots, 0), \dots$ ,  $P_j = (0, \dots, 0, \underbrace{1}_{j-1}, 0, \dots, 0)$ ,  $j = 5, \dots, k+2$ . Also by the property of  $k$ , we have  $P_1, P_2, P_3, P_5, \dots, P_{k+2}$  and arbitrary  $s-k-1$  points of  $X \setminus \alpha$  not on a linear  $(s-2)$ -space. We can put  $P_j = (0, \dots, 0, \underbrace{1}_{j-1}, 0, \dots, 0)$ ,  $j = k+3, \dots, s+2$ .

For every monomial  $M = X_1^{c_1} \cdots X_n^{c_n}$ ,  $c_1 + \dots + c_n = i$ ,  $i = 0, \dots, m-1$ . Put  $m_4 = m_{s+3} = m$ ,  $m_2 = m - i + c_1$ ,  $m_3 = m - i + c_2$ ,  $m_j = m - i + c_{j-2}$ ,  $j = 5, \dots, s+2$ . Put

$$t_1 = \max \left\{ m_j, \left\lfloor \frac{\sum_{l=2}^{k+2} m_l + k - 1}{k} \right\rfloor \mid j = 2, \dots, k+2 \right\}.$$

Since there always exists a linear  $(k-1)$ -space passing arbitrary  $k$  points of  $\{P_2, \dots, P_{k+2}\}$  and avoiding  $P_{i_0}$ , by Lemma 3.1 we can find  $t_1$  linear  $(k-1)$ -space avoiding  $P_{i_0}$ , say  $L_1, \dots, L_{t_1}$ , such that for every point  $P_j \in \{P_2, \dots, P_{k+2}\}$ , there are  $m_j$  linear  $(k-1)$ -spaces (including multiplicity) of  $\{L_1, \dots, L_{t_1}\}$  passing through the  $P_j$ .

Consider the set  $\{P_{k+3}, \dots, P_{s+3}\}$ . We remark that there is not any linear  $(s-k-1)$ -space containing  $P_{i_0}$  and  $s-k$  points of  $\{P_{k+3}, \dots, P_{s+3}\}$ . In fact, assume that there exists a linear  $(s-k-1)$ -space, say  $L'$ , containing  $P_{i_0}$  and  $s-k$  points of  $\{P_{k+3}, \dots, P_{s+3}\}$ . Then, linear  $(s-1)$ -space, say  $H$ , containing the linear  $(k-1)$ -space  $L_1$  and the linear  $(s-k-1)$ -space  $L'$  contains  $P_{i_0}$ . Therefore, this linear  $(s-1)$ -space  $H$  contains  $\alpha$ . So,  $H$  contains  $s+2$  points of  $X$ , a contradiction.

Put

$$t_2 = \max \left\{ m_j, \left\lfloor \frac{\sum_{l=k+3}^{s+3} m_l + s - k - 1}{s - k} \right\rfloor \mid j = k+3, \dots, s+3 \right\}.$$

Since there always exists a linear  $(s-k-1)$ -space passing arbitrary  $s-k$  points of  $\{P_{k+3}, \dots, P_{s+3}\}$ , by the above remark we can find  $t_2$  linear  $(s-k-1)$ -space avoiding  $P_{i_0}$ , say  $L'_1, \dots, L'_{t_2}$ , such that for every point  $P_j \in \{P_{k+3}, \dots, P_{s+3}\}$ , there are  $m_j$  linear  $(s-k-1)$ -spaces (including multiplicity) of  $\{L'_1, \dots, L'_{t_2}\}$  passing through the  $P_j$ . Put

$$t = \max\{t_1, t_2\}.$$

For  $j = 1, \dots, t$ , there always exists a linear  $(s-1)$ -space, say  $H_j$ , containing  $L_j$ ,  $L'_j$  and avoiding  $P_{i_0}$  (If  $H_j$  contains  $P_{i_0}$ , then  $H_j$  contains  $\alpha$  and  $L'_j$ . This implies that  $H_j$  contains  $s+2$  points of  $X$ , a contradiction). Since  $H_j$  contains  $L_j$  and  $L'_j$ ,  $j = 1, \dots, t$ , we get that for every point  $P_i \in \{P_2, \dots, P_{s+3}\}$ , there are  $m_i$  hyperplanes of  $\{H_1, \dots, H_t\}$  passing through the  $P_i$ . Therefore,

$$H_1 \cdots H_t \in \wp_2^{m_2} \cap \cdots \cap \wp_{s+3}^{m_{s+3}} = \wp_2^{m-i+c_1} \cap \wp_3^{m-i+c_2} \cap \wp_4^m \cap \wp_5^{m-i+c_3} \cap \cdots \cap \wp_{s+2}^{m-i+c_s} \cap \wp_{s+3}^m.$$

Moreover, since  $M \in \wp_2^{i-c_1} \cap \wp_3^{i-c_2} \cap \wp_5^{i-c_3} \cap \cdots \cap \wp_{s+2}^{i-c_s}$ , we get

$$H_1 \cdots H_t M \in \wp_2^m \cap \cdots \cap \wp_{s+3}^m = J.$$

By Remark 2.3 we get

$$\text{reg}(R/(J + \wp_{i_0}^m)) \leq \max\{t + i \mid i = 0, \dots, m-1\}.$$

We recall that

$$t = \max\{t_1, t_2\} = \max \left\{ m, \left\lfloor \frac{\sum_{l=2}^{k+2} m_l + k - 1}{k} \right\rfloor, \left\lfloor \frac{\sum_{l=k+3}^{s+3} m_l + s - k - 1}{s - k} \right\rfloor \right\}.$$

**Case 2.1.1:** If  $t = m$ , then

$$\max\{t + i \mid i = 0, \dots, m - 1\} \leq 2m - 1 \leq T_1.$$

**Case 2.1.2:** If  $t = \left\lfloor \frac{\sum_{l=2}^{k+2} m_l + k - 1}{k} \right\rfloor$ , then

$$\begin{aligned} t + i &= \left\lfloor \frac{\sum_{l=2}^{k+2} m_l + k - 1 + ki}{k} \right\rfloor = \left\lfloor \frac{(k+1)m - ki + \sum_{l=3}^{k+2} c_l + k - 1 + ki}{k} \right\rfloor \\ &\leq \left\lfloor \frac{(k+1)m + i + k - 1}{k} \right\rfloor \leq \left\lfloor \frac{(k+2)m + k - 2}{k} \right\rfloor = T_k. \end{aligned}$$

So,

$$\max\{t + i \mid i = 0, \dots, m - 1\} \leq T_k.$$

**Case 2.1.3:** If  $t = \left\lfloor \frac{\sum_{l=k+3}^{s+3} m_l + s - k - 1}{s - k} \right\rfloor$ , then

$$\begin{aligned} t + i &= \left\lfloor \frac{\sum_{l=k+3}^{s+3} m_l + s - k - 1 + (s - k)i}{s - k} \right\rfloor \\ &= \left\lfloor \frac{(s - k + 1)m - (s - k)i + \sum_{l=k+3}^{s+2} c_{l-2} + s - k - 1 + (s - k)i}{s - k} \right\rfloor \\ &= \left\lfloor \frac{(s - k + 1)m + \sum_{l=k+3}^{s+2} c_{l-2} + s - k - 1}{s - k} \right\rfloor. \end{aligned}$$

Note that  $\sum_{l=k+3}^{s+2} c_{l-2} \leq i \leq m - 1$ . Recall that we are considering Case 2.1:  $m = 1$  or  $s - k \geq k$  or  $s - k \geq 2$ .

**Case 2.1.3.1:** If  $m = 1$ , then we have

$$\begin{aligned} \max\{t + i \mid i = 0, \dots, m - 1\} &= \left\lfloor \frac{(s - k + 1)1 + s - k - 1}{s - k} \right\rfloor = 2 \\ &\leq \left\lfloor \frac{(s + 3)1 + s - 2}{s} \right\rfloor = T_s. \end{aligned}$$

**Case 2.1.3.2:** If  $s - k \geq k$ , then we have

$$\begin{aligned} \max\{t + i \mid i = 0, \dots, m - 1\} &= \left\lfloor \frac{(s - k + 1)m + \sum_{l=k+3}^{s+2} c_{l-2} + s - k - 1}{s - k} \right\rfloor \\ &\leq \left\lfloor \frac{(s - k + 1)m + m - 1 + s - k - 1}{s - k} \right\rfloor \\ &\leq \left\lfloor \frac{(s - k + 2)m + s - k - 2}{s - k} \right\rfloor \\ &\leq \left\lfloor \frac{(k + 2)m + k - 2}{k} \right\rfloor = T_k. \end{aligned}$$

The next step we need only consider case of  $s - k \geq 2$  and  $m \geq 2$  and  $s - k < k$ .

**Case 2.1.3.3:** If  $s - k \geq 3$  and  $m \geq 2$ , then we have

$$\begin{aligned} \max\{t + i \mid i = 0, \dots, m - 1\} &= \left\lfloor \frac{(s - k + 1)m + \sum_{l=k+3}^{s+2} c_{l-2} + s - k - 1}{s - k} \right\rfloor \\ &\leq \left\lfloor \frac{(s - k + 2)m + s - k - 2}{s - k} \right\rfloor \leq 2m - 1 \leq T_1. \end{aligned}$$



**Case 2.1.3.4:** If  $s - k = 2$  and  $s - k < k$  and  $m \geq 2$ , then since  $c_1 + \dots + c_s = i \leq m - 1$ , we have  $\sum_{l=k+3}^{s+2} c_{l-2} \leq m - 1$ . We consider two following for  $\sum_{l=k+3}^{s+2} c_{l-2}$ :

- If  $\sum_{l=k+3}^{s+2} c_{l-2} \leq m - 2$ , then we have

$$H_1 \cdots H_t M \in \wp_2^m \cap \cdots \cap \wp_{s+3}^m = J$$

with

$$\max\{t + i \mid i = 0, \dots, m - 1\} = \left\lfloor \frac{3m + \sum_{l=k+3}^{s+2} c_{l-2} + 1}{2} \right\rfloor \leq \left\lfloor \frac{4m - 1}{2} \right\rfloor = 2m - 1 \leq T_1.$$

- If  $\sum_{l=k+3}^{s+2} c_{l-2} = m - 1$ , then we have  $c_j = 0, j = 1, \dots, s - 2, i = m - 1$ . Therefore,  $m_2 = m_3 = m_5 = \dots = m_s = 1$  and

$$M \in \wp_2^{m-1} \cap \wp_3^{m-1} \cap \wp_5^{m-1} \cap \cdots \cap \wp_s^{m-1} \cap \wp_{s+1}^{m-1-c_{s-1}} \cap \wp_{s+2}^{m-1-c_s}.$$

We recall that the  $\beta$  in Case 2.1 is the linear  $(s - 1)$ -space containing  $P_4, \dots, P_{s+3}$  and avoiding  $P_{i_0}$ . Let  $K_1$  be the hyperplane containing  $\beta$  and avoiding  $P_{i_0}$ . Then, we have

$$K_1 M \in \wp_2^{m-1} \cap \wp_3^{m-1} \cap \wp_4 \cap \wp_5^m \cdots \cap \wp_s^m \cap \wp_{s+1}^{m-c_{s-1}} \cap \wp_{s+2}^{m-c_s} \cap \wp_{s+3}.$$

The linear  $(s - 1)$ -space, say  $\gamma_1$ , containing  $P_2, P_3, P_4, \dots, P_{s-1}, P_{s+1}, P_{s+3}$  avoids  $P_{i_0}$  (If  $P_{i_0} \in \gamma_1$ , then  $\gamma_1$  contains  $\alpha$ . So,  $\gamma_1$  is a linear  $(s - 1)$ -space containing  $s + 2$  points of  $X$ , a contradiction). Similarly, the linear  $(s - 1)$ -space, say  $\gamma_2$ , containing  $P_2, P_3, P_4, \dots, P_{s-1}, P_{s+2}, P_{s+3}$  avoids  $P_{i_0}$ . Let  $K_2$  be a hyperplane containing  $\gamma_1$  and avoiding  $P_{i_0}$ . Let  $K_3$  be a hyperplane containing  $\gamma_2$  and avoiding  $P_{i_0}$ . Then, we have

$$K_3^{c_s} K_2^{c_{s-1}} K_1 M \in \wp_2^m \cap \wp_3^m \cap \wp_4^{1+c_{s-1}+c_s} \cap \wp_5^m \cap \cdots \cap \wp_s^m \cap \wp_{s+1}^m \cap \wp_{s+2}^m \cap \wp_{s+3}^{1+c_{s-1}+c_s} = \wp_1^m \cap \cdots \cap \wp_{s+3}^m = J$$

with

$$c_s + c_{s-1} + 1 + i = 2m - 1.$$

So, in case of  $s - k = 2, s - k < k$  and  $m \geq 2$  by Remark 2.3 we get

$$\text{reg}(R/(J + \wp_{i_0}^m)) \leq 2m - 1 = T_1.$$

**Case 2.2:**  $m \geq 2$  and  $s - k = 1$  and  $s - k < k$ . Then  $\alpha$  contains  $P_1, \dots, P_{s+1}$  and there is not any linear  $(s - 2)$ -space containing  $s$  points of  $X$ . We consider two following cases:

**Case 2.2.1:** There is not any linear  $(s - 1)$ -space containing  $P_{s+2}, P_{s+3}$  and  $s - 1$  points of  $X \cap \alpha$ . So, in this case every linear  $(s - 1)$ -space containing arbitrary  $s$  points of  $\{P_1, \dots, P_{s+2}\}$  avoids  $P_{s+3}$ . Put  $P_{i_0} = P_{s+3} = (1, 0, \dots, 0)$ ,  $P_1 = (0, \underbrace{1}_2, 0, \dots, 0)$ , ...,  $P_j = (0, \dots, 0, \underbrace{1}_{j+1}, 0, \dots, 0)$ ,  $j = 1, \dots, s$ .

For every monomial  $M = X_1^{c_1} \cdots X_n^{c_n}$ ,  $c_1 + \dots + c_n = i$ ,  $i = 0, \dots, m - 1$ . Put  $m_j = m - i + c_j$ ,  $j = 1, \dots, s$ ,  $m_{s+1} = m_{s+2} = m$ . Put

$$t = \max \left\{ m_j, \left\lfloor \frac{\sum_{l=1}^{s+2} m_l + s - 1}{s} \right\rfloor \mid j = 1, \dots, s + 2 \right\}.$$

By Lemma 3.1 we can find  $t$  linear  $(s - 1)$ -space avoiding  $P_{i_0}$ , say  $L_1, \dots, L_t$ , such that for every point  $P_j \in \{P_1, \dots, P_{s+2}\}$ , there are  $m_j$  linear  $(s - 1)$ -spaces (including multiplicity) of  $\{L_1, \dots, L_t\}$  passing through the  $P_j$ .

For  $j = 1, \dots, t$ , let  $H_j$  be a hyperplane containing  $L_j$  and avoiding  $P_{i_0}$ . Then we have

$$H_1 \cdots H_t \in \wp_1^{m_1} \cap \cdots \cap \wp_{s+2}^{m_{s+2}} = \wp_1^{m-i+c_1} \cap \cdots \cap \wp_s^{m-i+c_s} \cap \wp_{s+1}^m \cap \wp_{s+2}^m.$$

Moreover, since  $M \in \wp_1^{i-c_1} \cap \cdots \cap \wp_s^{i-c_s}$ , we have

$$H_1 \cdots H_t M \in \wp_1^m \cap \cdots \cap \wp_{s+2}^m = J.$$

By Remark 2.3 we get

$$\operatorname{reg}(R/(J + \wp_{i_0}^m)) \leq \max\{t + i \mid i = 1, \dots, m-1\}.$$

If  $t = m$ , then

$$\max\{t + i \mid i = 1, \dots, m-1\} \leq 2m - 1 = T_1.$$

If  $t = \left\lceil \frac{\sum_{l=1}^{s+2} m_l + s - 1}{s} \right\rceil$ , then

$$\max\{t + i \mid i = 1, \dots, m-1\} \leq \left\lceil \frac{\sum_{l=1}^{s+3} m_l + s - 2}{s} \right\rceil = T_s.$$

**Case 2.2.2:** There is a linear  $(s-1)$ -space, say  $\beta$ , containing  $P_{s+2}$ ,  $P_{s+3}$  and  $s-1$  points of  $X \cap \alpha$ . We may assume that  $P_3, \dots, P_{s+2} \in \beta$ . Put  $P_{i_0} = P_{s+3}$  and  $J = \wp_1^m \cap \dots \cap \wp_{s+2}^m$ . If  $s = 2$ , then  $P_3, P_4, P_5$  lie on the line  $\beta$  and  $P_1, P_2 \notin \beta$ . Let  $Q_1$  be the linear  $(n-1)$ -space passing through  $P_3, P_1$  and avoiding  $P_5$ . Let  $Q_2$  be the linear  $(n-1)$ -space passing through  $P_4, P_2$  and avoiding  $P_5$ . Then

$$Q_1^m Q_2^m M \in J$$

for every monomial  $M = X_1^{c_1} \dots X_n^{c_n}$ ,  $c_1 + \dots + c_n = i$ ,  $i = 0, \dots, m-1$ . By Remark 2.3 we get

$$\operatorname{reg}(R/(J + \wp_{i_0}^m)) \leq \max\{2m + i \mid i = 1, \dots, m-1\} \leq 3m - 1 = T_1.$$

If  $s \geq 3$ , by Lemma 3.2 we get

$$\operatorname{reg}(R/(J + \wp_{i_0}^m)) \leq \max\{T_j \mid j = 1, \dots, n\}.$$

The proof of Proposition 3.3 is completed. □

The following proposition gives a sharp upper bound for the regularity index of  $s+3$  equimultiple fat points not on a linear  $(s-1)$ -space.

**Theorem 3.4.** *Let  $X = \{P_1, \dots, P_{s+3}\}$  be a set of distinct points not on a linear  $(s-1)$ -space in  $\mathbb{P}^n$ ,  $s \leq n$ , and  $m$  be a positive integer. Let*

$$Z = mP_1 + \dots + mP_{s+3}$$

*be the equimultiple fat points. Then,*

$$\operatorname{reg}(Z) \leq \max\{T_j \mid j = 1, \dots, n\},$$

*where*

$$T_j = \max \left\{ \left\lceil \frac{mq + j - 2}{j} \right\rceil \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a linear } j\text{-space} \right\}.$$

*Proof.* Let  $\wp_1, \dots, \wp_{s+3}$  be the homogeneous prime ideals of the polynomial ring  $R = K[X_0, \dots, X_n]$  corresponding to the points  $P_1, \dots, P_{s+3}$ . Put  $I = \wp_1^m \cap \dots \cap \wp_{s+3}^m$ . We have  $\operatorname{reg}(Z) = \operatorname{reg}(R/I)$ .

We argue by induction on  $s$ . For  $s = 1$ , the theorem is true by Lemma 2.4. We assume that the theorem is true for  $s-1$ . By Proposition 3.3, there exists a point  $P_{i_0} \in X$  such that

$$\operatorname{reg}(R/(J + \wp_{i_0}^m)) \leq \max\{T_j \mid j = 1, \dots, n\}, \quad (5)$$

where

$$J = \bigcap_{i \neq i_0} \wp_i^m.$$

Put  $Y = X \setminus \{P_{i_0}\}$ . Since  $X$  does not lie on a linear  $(s-1)$ -space, we have  $Y$  does not lie on a linear  $(s-2)$ -space. Put

$$T'_j = \max \left\{ \left\lceil \frac{m_{H \setminus \{P_{i_0}\}} + j - 2}{j} \right\rceil \mid H \text{ is a linear } j\text{-space} \right\}$$

with

$$m_{H \setminus \{P_{i_0}\}} = \sum_{P_i \in H \setminus \{P_{i_0}\}} m_i, \quad m_i = m.$$

By inductive assumption, we get

$$\text{reg}(R/J) \leq \max \{T'_j \mid j = 1, \dots, n\}.$$

We have  $T'_j \leq T_j$ ,  $j = 1, \dots, n$ . Thus,

$$\text{reg}(R/J) \leq \max \{T_j \mid j = 1, \dots, n\}. \quad (6)$$

By Lemma 2.1 we have

$$\text{reg}(R/I) = \max \{m-1, \text{reg}(R/J), \text{reg}(R/(J + \wp_{i_0}^m))\}. \quad (7)$$

Therefore, from (5), (6) and (7) we get

$$\text{reg}(R/I) \leq \max \{T_j \mid j = 1, \dots, n\}.$$

The proof of Theorem 3.4 is completed.  $\square$

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